

# Multi-Dimensional Cosmology and GUP

K. Zeynali<sup>1\*</sup>, F. Darabi<sup>2†</sup>, and H. Motavalli<sup>1‡</sup>

<sup>1</sup>*Department of Theoretical Physics and Astrophysics, University of Tabriz, 51666-16471, Tabriz, Iran.*

<sup>2</sup>*Department of Physics, Azarbaijan University of Shahid Madani, 53714-161, Tabriz, Iran.*

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## Abstract

We consider a multidimensional cosmological model with FRW type metric having 4-dimensional space-time and  $d$ -dimensional Ricci-flat internal space sectors with a higher dimensional cosmological constant. We study the classical cosmology in commutative and GUP cases and obtain the corresponding exact solutions for negative and positive cosmological constants. It is shown that for negative cosmological constant, the commutative and GUP cases result in finite size universes with smaller size and longer ages, and larger size and shorter age, respectively. For positive cosmological constant, the commutative and GUP cases result in infinite size universes having late time accelerating behavior in good agreement with current observations. The accelerating phase starts in the GUP case sooner than the commutative case. In both commutative and GUP cases, and for both negative and positive cosmological constants, the internal space is stabilized to the sub-Planck size, at least within the present age of the universe. Then, we study the quantum cosmology by deriving the Wheeler-DeWitt equation, and obtain the exact solutions in the commutative case and the perturbative solutions in GUP case, to first order in the GUP small parameter, for both negative and positive cosmological constants. It is shown that good correspondence exists between the classical and quantum solutions.

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## 1 Introduction

It is well known that quantum description of gravity must be considered when we want to deal with systems in the Planck scales, such as very early universe or strong gravitational field of a black hole. In the absence of gravity, quantum description of a system can be derived from classical consideration by replacing of Poisson bracket with usual commutation relations ( $\{ , \} \rightarrow \frac{1}{i\hbar} [ , ]$ ). When we want to consider the gravitational effect in quantum

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\*Email: k.zeinali@arums.ac.ir

†Email: f.darabi@azaruniv.edu (Corresponding author)

‡Email: motavalli@tabrizu.ac.ir

description of a system, some essential modification in the ordinary quantum principal are needed. General Uncertainty Principal (GUP) is a modification of Heisenberg Uncertainty Principal in the Planck scale. Such a generalization has already been considered in the context of string theory where the string can not probe distance smaller than the string size [1]-[6]. Some general view to the GUP and its application to cosmology are proposed in [7]-[13] and [14], [15], respectively. Michele Maggiore in the discussion of a Gedanken experiment for the measurement of the area of apparent horizon of a black hole in quantum gravity, using rather general and model independent consideration, showed that a minimum length of order of the Planck length in a Generalized Uncertainty Principal emerges naturally from any quantum theory of gravity [10]. In [7], the simplest form of the GUP in a one dimensional system, is written as:

$$\Delta x \Delta p \geq \frac{\hbar}{2} \left( 1 + \beta \frac{L_{Pl}^2}{\hbar^2} (\Delta p)^2 \right), \quad (1)$$

where  $L_{Pl} = \sqrt{\frac{G\hbar}{c^3}} = 10^{-33}$  is the Planck length and  $\beta$  is a positive constant to be of order unity. The new term in the above equation is important when  $x, \Delta x \approx L_{Pl}$ . It is possible to show that the above GUP relation (equation 1) can be derived from the following generalized Heisenberg algebra:

$$[x, p] = i\hbar(1 + \beta p^2). \quad (2)$$

Since cosmology provides the ground for testing physics at high energy, it seems natural to expect the effects of quantum gravity in this context. Alternatively, in cosmological systems, the scale factor, matter field and their conjugate momenta play the role of dynamical variables of the system; so, introducing GUP in the corresponding phase space is particularly relevant.

In the past few years the search for a consistent quantum theory of gravity and the quest for a unification of gravity with other forces have both led to a renewed interest in theories with extra spatial dimensions [18]-[19]. Extra spatial dimensions ideas date back to the seminal work of Kaluza [20] and Klein [21]. These extra spacial dimensions must be hidden and assumed to be unseen because they are compact and have small radius, presumably with typical dimensions of the order of planck length,  $O(10^{-33}cm)$ . At Planck time,  $t_p = \sqrt{\frac{\hbar G}{c^5}} = O(10^{-44}s)$ , the characteristic size of both internal and external dimensions are likely to have been the same, and the internal dimensions may have had a more direct role in the dynamics of evolution of the Universe.

As we discussed above, the existence of extra dimensions and the influence of GUP become evident at high energy. So, it is clear that in studying cosmology at high energy, we should presumably consider extra dimensions and GUP together [22], [23].

The paper is organized as follows. In section 2, we construct a multidimensional cosmology. By convenient coordinate transformation, we calculate it's Hamiltonian which describes an isotropic oscillator-ghost-oscillator system. In section 3, we investigate the equations of motion in classical model, at first in commutative case and then by considering the influence of GUP. Section 4, devotes to quantum cosmology. We construct the

Wheeler-DeWitt (WD) equation in commutative and GUP cases and solve it to obtain the wavefunction of the corresponding universes.

## 2 The Cosmological Model

It is well known that universe in large scale is homogenous and isotropic and has 3-dimensional space at least in the order of experimental tests. So it is possible to exist unseen compact internal space (Extra dimension) with small radius. For our investigations, we consider a multi-dimensional cosmological model in which the space-time is established by a FRW type metric of 4-dimensional space time and a  $d$ -dimensional Ricci-flat internal space [24]:

$$ds^2 = -dt^2 + \frac{R^2(t)}{(1 + \frac{k}{4}r^2)}(dr^2 + r^2 d\Omega^2) + a^2(t)g_{ij}^{(d)} dx^i dx^j, \quad (3)$$

where  $k = 1, 0, -1$  represents the usual spatial curvature of external space in FRW model,  $R(t)$  and  $a(t)$  are the scale factors of the external and internal space respectively, and  $g_{ij}^{(d)}$  is the metric of the internal space which is assumed to be Ricci-flat. The total number of dimensions is  $D = 3 + d$ . The Ricci scalar can be derived from the metric(3) [24]

$$\mathcal{R} = 6 \left( \frac{\ddot{R}}{R} + \frac{k + \dot{R}^2}{R^2} \right) + 2d \frac{\ddot{a}}{a} + d(d-1) \left( \frac{\dot{a}}{a} \right)^2 + 6d \frac{\dot{a}\dot{R}}{aR}, \quad (4)$$

where a dot represents differentiation with respect to  $t$ . We consider an Einstein-Hilbert action functional with a  $D$ -dimensional cosmological constant  $\Lambda$ :

$$\mathcal{S} = \frac{1}{2k_D^2} \int_M d^D x \sqrt{-g} (\mathcal{R} - 2\Lambda) + \mathcal{S}_{YGH}, \quad (5)$$

where  $k_D$  is the  $D$ -dimensional gravitational constant and  $\mathcal{S}_{YGH}$  is the York-Gibbons-Hawking boundary term. Substitution Eq.(4), after dimensional reduction we have:

$$\mathcal{S} = -v_{D-1} \int dt \left\{ 6\dot{R}^2 \Phi R + 6\dot{R}\dot{\Phi} R^2 + \frac{d-1}{d} \frac{\dot{\Phi}^2}{\Phi} R^3 - 6k\Phi R + 2\Phi R^3 \Lambda \right\}, \quad (6)$$

where

$$\Phi = \left( \frac{a}{a_0} \right)^d, \quad (7)$$

and  $a_0$  is the compactification scale of the internal space at present time. We can set  $v_{D-1} = 1$ . To make the Lagrangian suitable, we consider the following change of variables<sup>1</sup>.

$$\Phi R^3 = \Upsilon^2 (x_1^2 - x_2^2), \quad (8)$$

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<sup>1</sup>Here we have some differences from [24] in defining the parameters of equations (10).

$$\begin{aligned}\Phi^{\rho+} R^{\sigma-} &= \Upsilon(x_1 + x_2), \\ \Phi^{\rho-} R^{\sigma+} &= \Upsilon(x_1 - x_2).\end{aligned}\tag{9}$$

with

$$\begin{aligned}\rho_{\pm} &= \frac{1}{2} \pm \frac{1}{2} \sqrt{\frac{3}{d(d+2)}}, \\ \sigma_{\pm} &= \frac{3}{2} \pm \frac{1}{2} \sqrt{\frac{3d}{d+2}}, \\ \Upsilon &= \frac{1}{2} \sqrt{\frac{d+3}{d+2}}.\end{aligned}\tag{10}$$

where  $R = R(x_1, x_2)$  and  $\Phi = \Phi(x_1, x_2)$  are functions of new variables  $x_1, x_2$ . Using the above transformations and concentrating on  $k = 0$ , the Lagrangian becomes

$$\mathcal{L} = (\dot{x}_1^2 - \dot{x}_2^2) + \frac{\Lambda}{2} \left( \frac{d+3}{d+2} \right) (x_1^2 - x_2^2).\tag{11}$$

We can write the effective Hamiltonian as <sup>2</sup>

$$\mathcal{H} = \left( \frac{p_1^2}{4} + \omega^2 x_1^2 \right) - \left( \frac{p_2^2}{4} + \omega^2 x_2^2 \right),\tag{12}$$

where

$$\omega^2 = -\frac{1}{2} \left( \frac{d+3}{d+2} \right) \Lambda.\tag{13}$$

Equations (12), (13) and (8) show that the potential energy for our oscillator-ghost-oscillator system [29] is proportional to vacuum energy  $\Lambda$  times the volume of the multidimensional universe and the total momentum of such a system is  $p_{tot}^2 = \frac{p_1^2}{2} - \frac{p_2^2}{2}$ .

The effective oscillators in Eq.(12) appear to be decoupled and the dynamics of two dynamical variables, namely  $x_1$  and  $x_2$  seems to be independent of each other. However, the Hamiltonian constraint ( $\mathcal{H} = 0$ ) imposing on oscillators connects their dynamics <sup>3</sup>. Therefore, the dynamics of external large dimensions  $R$  and internal compactified extra dimensions  $a$  are inevitably related to each other by the equation (9).

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<sup>2</sup> In the Hamiltonian of cosmological models having matter contents, usually a wrong sign between geometric and matter sectors appears. This is due to the fact that the matter content of the universe has positive energy while the gravitational content has negative energy. In the present model, there is no explicit matter content, however, we may interpret the sector of extra compact dimensions as a fictitious matter field. Then, the Hamiltonian of the variables  $R, a$  having a sign difference between the four and extra dimensional sectors is transformed to that of the variables  $x_1, x_2$  with the same sign difference between  $x_1$  and  $x_2$  parts. These Hamiltonians are usually called oscillator-ghost-oscillator.

<sup>3</sup>We know that general relativity is a *time reparametrization invariant* theory. Every theory which is diffeomorphism invariant casts into the constraint systems. Therefore, general relativity is a constraint system whose constraint is the zero energy condition ( $\mathcal{H} = 0$ ).

### 3 Classical solutions

#### 3.1 Commutative case

As in [24, 29], the dynamical variable defined in (9) and their conjugate momenta satisfy:

$$\{x_\mu, p_\nu\}_P = \eta_{\mu\nu}, \quad (14)$$

where  $\eta_{\mu\nu}$  is the two dimensional Minkowski metric and  $\{ , \}_P$  represents the Poisson bracket. The equations of motion can be written as:

$$\begin{aligned} \dot{x}_\mu &= \{x_\mu, \mathcal{H}\}_P = \frac{1}{2}p_\mu, \\ \dot{p}_\mu &= \{p_\mu, \mathcal{H}\}_P = -2\omega^2 x_\mu, \end{aligned} \quad (15)$$

Combining two equations in (15) we obtain

$$\ddot{x}_\mu + \omega^2 x_\mu = 0. \quad (16)$$

First, we assume a negative cosmological constant. According to (13),  $\omega^2$  is then positive. Considering (12), it is obvious that Eq.(16) describes the equations of two ordinary uncoupled harmonic oscillators whose solutions are

$$x_\mu(t) = A_\mu e^{i\omega t} + B_\mu e^{-i\omega t}, \quad (17)$$

where  $A_\mu$  and  $B_\mu$  are constants of integration. Imposing the Hamiltonian constraint ( $\mathcal{H} = 0$ ) introduce the following relation on these constants.

$$A_\mu B^\mu = 0. \quad (18)$$

Finally, using (7) and (9), the scale factors take the following forms

$$\begin{aligned} a(t) &= k_1 [\sin(\omega t + \phi_1)]^{\frac{\sigma_+}{d(\rho_+ \sigma_+ - \rho_- \sigma_-)}} [\sin(\omega t + \phi_2)]^{\frac{-\sigma_-}{d(\rho_+ \sigma_+ - \rho_- \sigma_-)}}, \\ R(t) &= k_2 [\sin(\omega t + \phi_1)]^{\frac{-\rho_-}{\rho_+ \sigma_+ - \rho_- \sigma_-}} [\sin(\omega t + \phi_2)]^{\frac{\rho_+}{\rho_+ \sigma_+ - \rho_- \sigma_-}}, \end{aligned} \quad (19)$$

where  $k_1$  and  $k_2$  are arbitrary constants and  $\phi_1$  and  $\phi_2$  are arbitrary phases. If we consider the Hamiltonian constraint, the following relation is imposed on these constants

$$\frac{4(d+2)}{d+3} k_1^d k_2^3 \cos(\phi_1 - \phi_2) = 0. \quad (20)$$

Because of  $k_1, k_2 \neq 0$ , this leads to  $\phi_1 - \phi_2 = \frac{\pi}{2}$ . In what follows, we investigate the behavior of a universe with one internal dimension ( $D = 3 + 1$ ).

By setting  $\phi_1 = \frac{\pi}{2}$  and  $\phi_2 = 0$ , we find:

$$\begin{aligned} R(t) &= k_2 \sqrt{\sin(\omega t)}, \\ a(t) &= k_1 \frac{\cos(\omega t)}{\sqrt{\sin(\omega t)}}, \end{aligned} \quad (21)$$

whose classical loci is given by

$$a = \frac{1}{R} \sqrt{1 - R^4}, \quad (22)$$

where the values of  $k_1$  and  $k_2$  are absorbed in the values of  $R$  and  $a$ . Figure.2 (left), shows this loci for negative cosmological constant. It is seen that the dynamics of  $R$  is constrained by the dynamics of  $a$  and vice versa. This is expectable, because the dynamics of two variables is subject to the zero energy condition so called Hamiltonian constraint. Using these solutions, we can calculate the Hubble and deceleration parameter for both  $R(t)$  and  $a(t)$  as

$$\begin{aligned} H_R(t) &= \frac{\dot{R}(t)}{R(t)} = \frac{\omega}{2} \cot(\omega t), \\ q_R(t) &= -\frac{R(t)\ddot{R}(t)}{\dot{R}^2(t)} = 1 + 2 \tan^2(\omega t), \\ H_a(t) &= \frac{\dot{a}(t)}{a(t)} = -\frac{\omega}{2} (\cot(\omega t) + 2 \tan(\omega t)), \\ q_a(t) &= -\frac{a(t)\ddot{a}(t)}{\dot{a}^2(t)} = -\frac{2 \cos^2(\omega t)(5 + \cos(2\omega t))}{(-3 + \cos(2\omega t))^2}. \end{aligned} \quad (23)$$

From (21), we see that as time increases from  $t = 0$  toward  $\frac{\pi}{2\omega}$ ,  $R(t)$  and  $a(t)$  are increasing and decreasing functions of  $t$ , toward the maximum and zero, respectively. At  $t = \frac{\pi}{2\omega}$ ,  $R(t)$  reaches to its maximum and  $a(t)$  is contracted toward zero. From  $t = \frac{\pi}{2\omega}$  to  $t = \frac{\pi}{\omega}$ ,  $R(t)$  is contracted toward zero while  $a(t)$  is decreasing in the region of negative values. Although the negative value for  $a(t)$  is meaningless, but it is  $a^2(t) > 0$  which is physically viable in the definition of metric<sup>4</sup>. Therefore, we stick to the behavior of  $a^2(t) > 0$  instead of  $a(t)$ . As is seen in Fig.1 (solid lines), from  $t = \frac{\pi}{2\omega}$  to  $t = \frac{\pi}{\omega}$ , both  $R^2(t)$  and  $a^2(t)$  become decreasing and increasing functions, respectively. According to the behavior mentioned above, the four dimensional sector begins from a Big Bang at  $t \simeq 0$ , then expands till  $t = \frac{\pi}{2\omega}$  toward a maximum value after which the gravity takes over the expansion and this sector starts contracting toward a big crunch at  $t \simeq \frac{\pi}{\omega}$ . In order to avoid of initial singularity, we will assume the Big Bang to occur at the Planck time  $t_{Pl}$ . In fact, it is reasonable that at planck time we set  $R(t_{Pl}) = a(t_{Pl})$ . In this regard, the extra dimensional sector begins from  $a^2(t_{Pl})$ , then contracts till  $t = \frac{\pi}{2\omega}$  toward a vanishing minimum value after which this sector starts expanding toward  $a^2(\frac{\pi}{\omega} - t_{Pl}) = a^2(t_{Pl})$ . we see that during the whole time evolution of universe ( $t_{Pl} \leq t \leq \frac{\pi}{\omega} - t_{Pl}$ ), the scale factor of internal space is contracted toward zero and can never exceed  $a(t_{Pl})$ . By considering present value of Hubble constant, we see that the age of universe is  $t_{present} = \frac{1}{\omega} \cot^{-1}(\frac{2H_0}{\omega})$  whose order of magnitude is in agreement with present observations, namely  $\omega^{-1} \approx 10^{17} s$ . So, the present universe is in tideway to get to the maximum and minimum of  $R^2(t)$  and  $a^2(t)$ , respectively, within  $\Delta t \approx 0.57\omega^{-1}$ .

In the time interval  $\frac{\pi}{\omega} \leq t \leq \frac{2\pi}{\omega}$ ,  $R^2(t)$  and  $a^2(t)$  become negative Figs.1 (solid lines). To avoid this problem we may consider two options:

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<sup>4</sup>This situation is similar to that of wavefunction in quantum mechanics, where it is the square of the wavefunction which is physical and not the wavefunction itself.

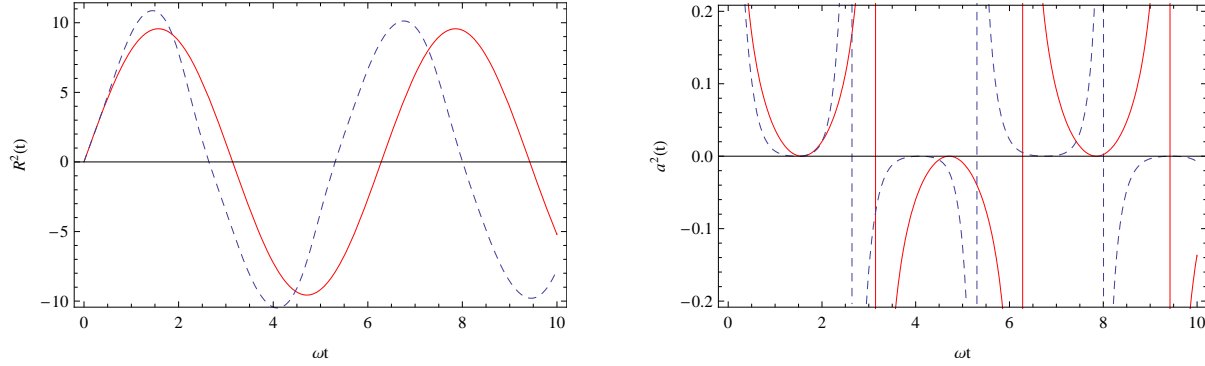


Figure 1: Time evolution of the (squared) scale factors of universe with one extra dimension and negative cosmological constant. Solid and dashed lines refer to the scale factors in commutative and GUP framework respectively. Left and right figures are the external and internal dimensions respectively.

1) Classically, negative values for  $R^2(t)$  and  $a^2(t)$  are nonphysical, so we may leave the cosmology with imaginary scale factors as the nonphysical solutions of the Einstein equations and think that the physical universe will end at  $t = \frac{\pi}{\omega}$ , with no further extension or history.

2) Quantum mechanically, as the scale factor  $R(t)$  goes to zero (when  $t$  approaches the big crunch singularity), the strong quantum gravitational effects can drastically change the initial conditions so that  $R^2(t)$  and  $a^2(t)$  may become again nonzero positive functions capable of establishing another Big Bang.

By, considering

$$\begin{aligned} R(t_{Pl}) &= k_2 \sqrt{\sin(\omega t_{Pl})}, \\ a(t_{Pl}) &= k_1 \frac{\cos(\omega t_{Pl})}{\sqrt{\sin(\omega t_{Pl})}}, \end{aligned} \quad (24)$$

the initial condition  $R(t_{Pl}) = a(t_{Pl})$  results in

$$\frac{k_2}{k_1} = 10^{61}, \quad (25)$$

by which we obtain the following ratio

$$\frac{R(t)}{a(t)} = 10^{61} \tan(\omega t). \quad (26)$$

If we consider the present radius of external space to be equal to the radius of observed universe, namely  $10^{28} \text{cm}$ , then we see that the present radius of internal space is about the Planck length ( $10^{-33} \text{cm}$ ).

By considering smallness of the internal dimension and using (23) to evaluate the approximate magnitude of  $H_a(t_{\text{present}}) \sim \omega \approx 10^{-17}$ , we see that the variation of  $a(t)$  at present time is ignorable. So, in the context of negative cosmological constant, the compactification and the stabilization of the internal space at present status of the universe is established.

Now, we assume a positive cosmological constant for which  $\omega^2$  is negative. If we replace  $\omega^2$  with  $-\omega^2$  in Eq.(16), the new solutions are then obtained by replacing trigonometric

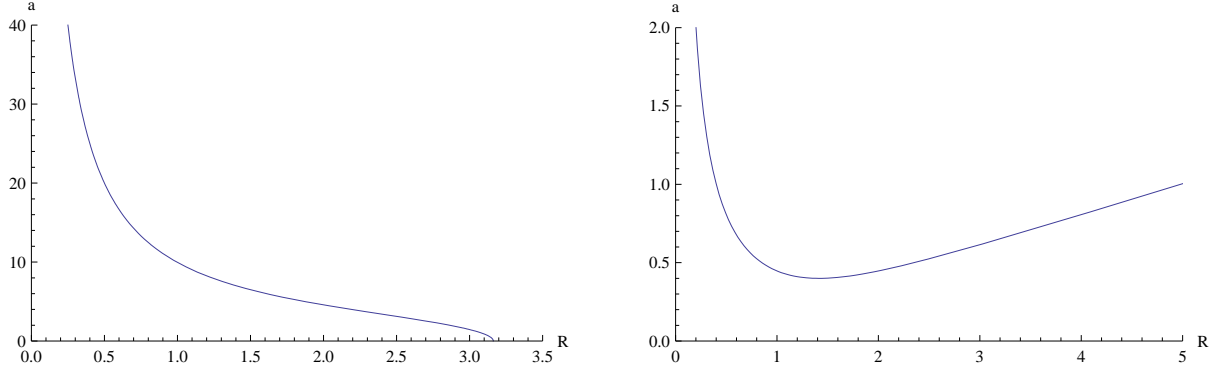


Figure 2: Left figure is classical loci in the negative cosmological context and right figure is the classical loci in the positive cosmological context.

functions by their hyperbolic counterparts in the solutions (19). By considering Hamiltonian constrain we obtain the following solutions

$$\begin{aligned} a(t) &= k_1 [\cosh(\omega t)]^{\frac{\sigma_+}{d(\rho_+ \sigma_+ - \rho_- \sigma_-)}} [\sinh(\omega t)]^{\frac{-\sigma_-}{d(\rho_+ \sigma_+ - \rho_- \sigma_-)}}, \\ R(t) &= k_2 [\cosh(\omega t)]^{\frac{-\rho_-}{\rho_+ \sigma_+ - \rho_- \sigma_-}} [\sinh(\omega t)]^{\frac{\rho_+}{\rho_+ \sigma_+ - \rho_- \sigma_-}}, \end{aligned} \quad (27)$$

where for  $d = 1$  we have

$$\begin{aligned} a(t) &= k_1 \frac{\cosh(\omega t)}{\sqrt{\sinh(\omega t)}}, \\ R(t) &= k_2 \sqrt{\sinh(\omega t)}, \end{aligned} \quad (28)$$

whose classical loci is given by

$$a = \frac{1}{R} \sqrt{1 + R^4}. \quad (29)$$

Figure.2 (right), shows this loci for positive cosmological constant. As in the negative cosmological constant case, the dynamics of both  $R$  and  $a$  is constrained by the Hamiltonian constraint. The scale factor ratio is obtained

$$\frac{R(t)}{a(t)} = 10^{61} \tanh(\omega t), \quad (30)$$

and we have

$$\begin{aligned} H_R(t) &= \frac{\dot{R}(t)}{R(t)} = \frac{\omega}{2} \coth(\omega t), \\ q_R(t) &= -\frac{R(t) \ddot{R}(t)}{\dot{R}^2(t)} = 1 - 2 \tanh^2(\omega t), \\ H_a(t) &= \frac{\dot{a}(t)}{a(t)} = \frac{\omega}{2} (-\coth(\omega t) + 2 \tanh(\omega t)), \\ q_a(t) &= -\frac{a(t) \ddot{a}(t)}{\dot{a}^2(t)} = -\frac{2 \cosh^2(\omega t) (5 + \cosh(2\omega t))}{(-3 + \cosh(2\omega t))^2}. \end{aligned} \quad (31)$$



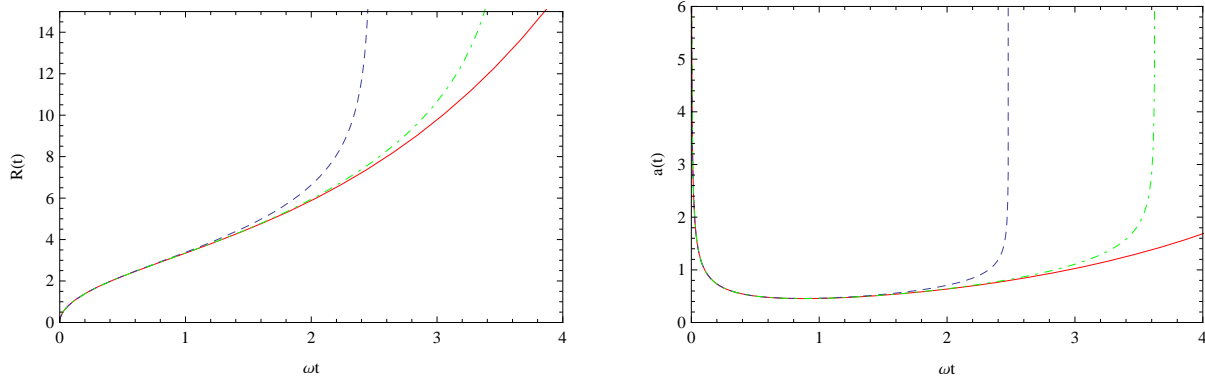


Figure 3: Time evolution of the scale factors of universe with one extra dimension and positive cosmological constant. Solid lines refer to the scale factors in commutative case, dashed and dot-dashed lines refer to the scale factors in GUP framework with  $\beta = 10^{-4}$  and  $\beta = 10^{-5}$  respectively. Left and right figures are the external and internal dimensions respectively.

Note that, like the case of negative cosmological constant, the magnitude of the radius of external to internal space is asymptotically ( $t \rightarrow \infty$ ) about  $10^{61}$ . Equations (28) show that  $R(t)$  is an increasing function of time. But,  $a(t)$  at first decrease with time till  $t \simeq 0.88\omega^{-1}$  and then increase exponentially (see Fig. 3). As we mentioned above, at planck time, the characteristic size of both internal and external dimensions are assumed to be the same, namely ( $R(t_{Pl}) = a(t_{Pl})$ ). If we consider the age of universe about  $\omega^{-1} \simeq 10^{17}s$ , we see that at present time we are around the minimum point of  $a(t)$ . Also by some calculation on  $a(t)$ , we find that in the time interval  $t_{Pl} \leq t \leq 141\omega^{-1}$ ,  $a(t)$  can never exceed it's initial value at Planck time  $a(t_{Pl})$ . Therefore, the internal scale factor remains small for a very long period with a duration of about 140 times of the present age of the universe.

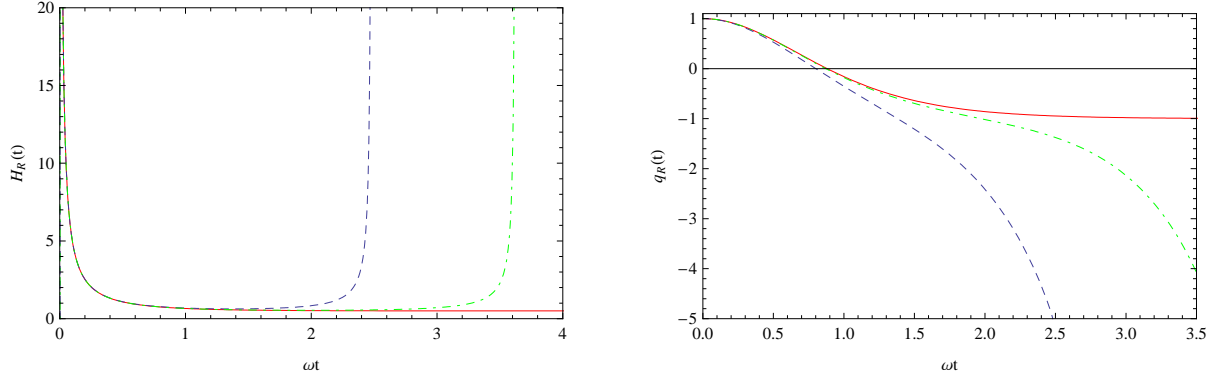


Figure 4: Left and right figures are respectively Hubble an deceleration parameters of external space for universe with one extra dimension and positive cosmological constant. Solid lines refer to the scale factors in commutative case, dashed and dot-dashed lines refer to the scale factors in GUP framework with  $\beta = 10^{-4}$  and  $\beta = 10^{-5}$  respectively.

### 3.2 GUP case

In more than one dimension, it can be shown that the generalized Heisenberg algebra corresponding to GUP is defined by the following commutation relations [7, 8]

$$[x_i, p_j] = i(\delta_{ij} + \beta \delta_{ij} p^2 + \beta' p_i p_j), \quad (32)$$

where  $p^2 = \sum p_i p_i$  and  $\beta \beta' > 0$  are considered as small quantities of first order. Throughout the whole process we work in the units with  $\hbar = G = c = 1$ . We assume that momenta commute with momenta

$$[p_i, p_j] = 0. \quad (33)$$

Using Jacobi identity  $[[x_i, x_j], p_k] + [[x_j, p_k], x_i] + [[p_k, x_i], x_j] = 0$  the commutation relation for the coordinates are obtained as:

$$[x_i, x_j] = i \frac{(2\beta - \beta') + (2\beta + \beta')\beta p^2}{(1 + \beta p^2)} (p_i x_j - p_j x_i). \quad (34)$$

It is clear from the above equation that the coordinates do not commute. So, we can not work in the position space to construct Hilbert space representations. But, if we choose the special case  $\beta' = 2\beta$ , we can see from equation (34) that coordinates commute to first order in  $\beta$  and thus we can work in coordinate representation. By the following definitions, equations (32) and (33) with  $\beta' = 2\beta$  can be realized up to first order in  $\beta$  (see Appendix)[30]

$$x_i = x_{i0}, \quad p_i = p_{i0}(1 + \beta p_0^2), \quad (35)$$

where  $[x_{i0}, p_{j0}] = i\delta_{ij}$ ,  $p_0^2 = \sum p_{i0} p_{i0}$  and  $p_{i0} = -i \frac{\partial}{\partial x_{i0}}$ . We can show that the  $p^2$  term in any Hamiltonian can be written as [30]

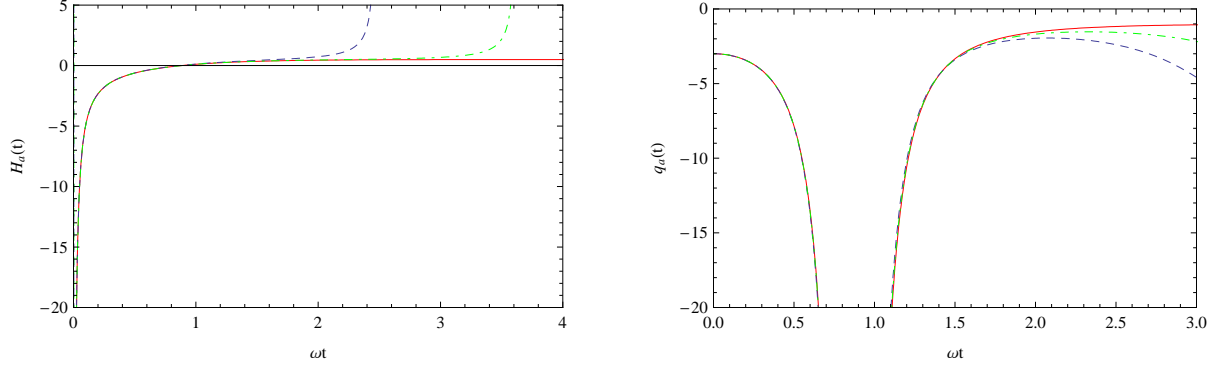


Figure 5: Left and right figures are respectively hubble and deceleration parameters of internal space for universe with one extra dimension and positive cosmological constant. Solid lines refer to the scale factors in commutative case, dashed and dot-dashed lines refer to the scale factors in GUP framework with  $\beta = 10^{-4}$  and  $\beta = 10^{-5}$  respectively.

$$p^2 = p_0^2 + 2\beta p_0^4. \quad (36)$$

In our model, by considering  $\beta' = 2\beta$  and in first order in  $\beta$ , the commutation relation between position and momentum operators can be summarized as

$$[x_1, p_1] = i(1 + \beta p^2 + 2\beta p_1^2), \quad [x_2, p_2] = i(1 + \beta p^2 + 2\beta p_2^2), \quad (37)$$

$$[x_1, p_2] = [x_2, p_1] = 2i\beta p_1 p_2, \quad (38)$$

$$[x_i, x_j] = [p_i, p_j] = 0, \quad i, j = 1, 2. \quad (39)$$

where  $p_{tot}^2 = \frac{1}{2}(p_1^2 + p_2^2)$ . We aim to investigate the effects of the classical version of GUP. As we know, we must replace the quantum mechanical commutators with the classical poisson bracket as  $[P, Q] \rightarrow i\{P, Q\}$ . Thus, in classical phase space the GUP deformed poisson algebra is achieved from (37)-(39) by considering  $[P, Q] \rightarrow i\{P, Q\}$ .

$$\{x_1, p_1\} = (1 + \beta p^2 + 2\beta p_1^2), \quad \{x_2, p_2\} = (1 + \beta p^2 + 2\beta p_2^2), \quad (40)$$

$$\{x_1, p_2\} = \{x_2, p_1\} = 2\beta p_1 p_2, \quad (41)$$

$$\{x_i, x_j\} = \{p_i, p_j\} = 0, \quad i, j = 1, 2, \quad (42)$$

In [31] such a deformation algebra is used on classical orbit of particles in a central force field and on kepler third law. It is notable that this modification is significant at Plank scale

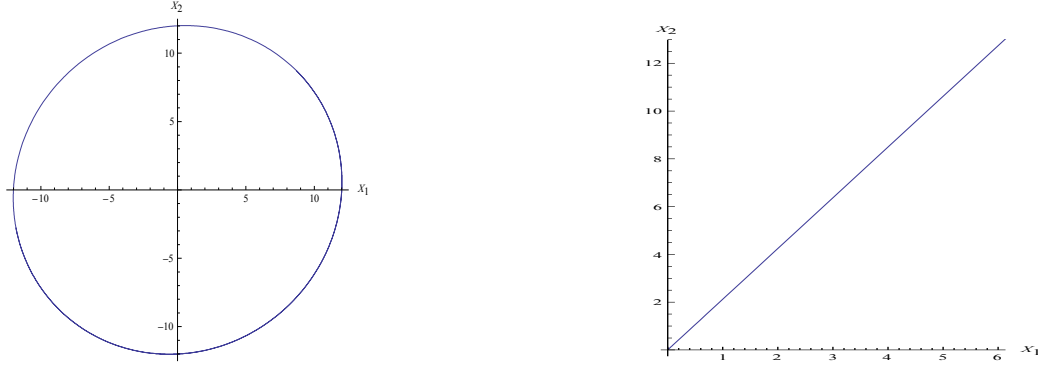


Figure 6: Left and right figures show the classical trajectories in  $x_1 - x_2$  plane for a universe with one extra dimension in the context of negative and positive cosmological constant, respectively.

and we need quantum description. But, before quantizing the model we want to construct a deformed classical cosmology. Note that, in transition from quantum commutation relations to Poisson brackets we shall keep the GUP's parameter  $\beta$  fixed as  $\hbar \rightarrow 0$ . The equations of motion can be written as (see [32])

$$\dot{x}_1 = \{x_1, H\} = \frac{1}{2}p_1(1 + 5\beta p^2), \quad (43)$$

$$\dot{x}_2 = \{x_2, H\} = -\frac{1}{2}p_2(1 - 3\beta p^2), \quad (44)$$

$$\dot{p}_1 = \{p_1, H\} = -2\omega^2 x_1(1 + \beta p^2 + 2\beta p_1^2) + 4\omega^2 \beta x_2 p_1 p_2, \quad (45)$$

$$\dot{p}_2 = \{p_2, H\} = 2\omega^2 x_2(1 + \beta p^2 + 2\beta p_2^2) - 4\omega^2 \beta x_1 p_1 p_2. \quad (46)$$

These show that the deformed classical cosmology forms a system of nonlinear coupled differential equations that can not be solved analytically.

In the context of negative cosmological constant,  $\omega^2$  is positive (13). Numerical solutions of Eqs. (43)-(46) shows that  $R(t)$  and  $a(t)$  like the classical ones have periodic behavior, but the time interval between Big Bang and big crunch is shortened (dashed lines, Fig.1). For later period, the time interval becomes longer with respect to first one while the maximum value of  $R(t)$  is shorter. Comparison of the results in the present GUP framework with the previous classical commutative case (Fig. 1) reveals that in the present framework we have larger external and smaller internal dimensions. Also the rate of variation of  $a(t)$  is less than the previous case. So, the compactification and the stabilization of internal space in GUP framework is more favored than the classical commutative case.

By replacing  $\omega^2$  with  $-\omega^2$  in Eqs.(43)-(46), we can get the corresponding equations in the case of positive cosmological constant. By numerical analysis, we find that at early time the scale factors behave like classical commutative case (Fig. 3), but at late time both the external and internal dimensions are greater than those of commutative case. Also, investigation in time dependence of the Hubble and deceleration parameters shows that at

early time these parameters nearly behave like classical commutative case, but at late time they behave very differently.

Some recent observations indicate that the universe is currently undergoing an accelerating period of expansion. This is consistent with the results obtained here considering a positive cosmological constant. Depiction of the Hubble and deceleration parameters for the scale factors  $R$  and  $a$  can be of useful help to understand the origin of current acceleration in this multidimensional model. To this end, we have depicted  $H_R$ ,  $q_R$  and  $H_a$ ,  $q_a$  within the figures 4 and 5 (see solid lines). Figure 4 shows that  $q_R$  becomes negative a little bit earlier than  $\omega t \sim 1$ , namely the present age of the universe. This means, the acceleration of the universe has started recently in the present multidimensional commutative case. On the other hand, Fig.5 shows that  $q_a$  is always negative and has a minimum at the position where  $q_R$  becomes negative. Therefore, it seems the behavior of  $q_a$  is responsible for the behavior of  $q_R$  and vice versa, as is interpreted in the following. The present model describes a multidimensional cosmology, with a positive cosmological constant. Typically, a standard 4-dimensional FRW cosmology with a positive cosmological constant predicts an accelerating behavior of the scale factor. In a multidimensional cosmology, however, it is reasonable to think that the overall repulsive force due to the positive cosmological constant manifests as an interplay between the accelerating and decelerating behaviors of the external and internal scale factors. Looking at the figures 4 and 5 reveals that, typically for both commutative and GUP cases, at the beginning of time,  $q_R$  is positive ( $R$  is decelerating) while  $q_a$  is negative ( $a$  is accelerating). As time passes,  $q_R$  approaches the threshold of negative values ( $R$  is less decelerating) while  $q_a$  goes to more negative values ( $a$  is highly accelerating). When  $q_R$  enters the region of negative values ( $R$  is accelerating) the  $q_a$  reaches its minimum ( $a$  stops the increasing acceleration; the minimum is not shown in Fig.5). Finally,  $q_R$  becomes more negative ( $R$  is highly accelerating) and  $q_a$  goes to rather less negative values ( $a$  is slowly accelerating). In order to better compare the results obtained in commutative and GUP cases, we have depicted all the GUP diagrams for two values of the parameter  $\beta$  with a difference of ten times order of magnitude. It is seen that as the value of  $\beta$  becomes smaller, the results of GUP more coincide with the results of the commutative case. Considering a very small value for the parameter  $\beta$ , for example suggested by string theory, reveals that at least at the present age of the universe ( $t \simeq \omega^{-1}$ ) it is impossible to distinguish between the GUP and commutative cases in that which approach fits better the data.

## 4 Quantum solutions

### 4.1 Commutative case

To investigate the quantum version of the model, we use the WD equation,  $\mathcal{H}\Psi = 0$ . Here  $\mathcal{H}$  is the operator form of the Hamiltonian (12). Using the canonical procedure to quantum mechanics that the phase space variables are replaced with quantum operators, by replacing  $p_i = -i\frac{\partial}{\partial x_i}$  in (12) we get the following WD equation:

$$\left[ -\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + 4\omega^2(x_1^2 - x_2^2) \right] \Psi(x_1, x_2) = 0. \quad (47)$$

This equation is the quantum version of oscillator-ghost oscillator system with zero energy condition. We can solve it by Formal variable separation approach as:

$$\Psi_{n_1, n_2}(x_1, x_2) = U_{n_1}(x_1)V_{n_2}(x_2), \quad (48)$$

$$\frac{\partial^2 W_i}{\partial x_i^2} + (\lambda - 4\omega^2 x_i^2)W_i = 0, \quad W_i(i = 1, 2) = U, V \quad (49)$$

in negative cosmological background, the solutions are

$$U_{n_1}(x_1) = \left(\frac{2\omega}{\pi}\right)^{1/4} \frac{e^{-\omega x_1^2}}{\sqrt{2^{n_1} n_1!}} H_{n_1}(\sqrt{2\omega} x_1), \quad (50)$$

$$V_{n_2}(x_2) = \left(\frac{2\omega}{\pi}\right)^{1/4} \frac{e^{-\omega x_2^2}}{\sqrt{2^{n_2} n_2!}} H_{n_2}(\sqrt{2\omega} x_2), \quad (51)$$

with  $\lambda = 2n_i + 1$  and the restriction  $n_1 = n_2 = n$ . Here,  $H_n(x)$  are Hermite polynomial and the eigenfunction are normalized as:

$$\int_{-\infty}^{+\infty} e^{-x^2} H_n(x) H_m(x) dx = 2^n \pi^{1/2} n! \delta_{nm}. \quad (52)$$

The general solution of the WD equation can be written as a superposition of the above eigenfunctions:

$$\Psi(x_1, x_2) = \left(\frac{2\omega}{\pi}\right)^{1/2} e^{-\omega(x_1^2 + x_2^2)} \sum_n \frac{c_n}{2^n n!} H_n(\sqrt{2\omega} x_1) H_n(\sqrt{2\omega} x_2). \quad (53)$$

Coefficients  $c_n$  are so chosen as to make the states coherent [33]:

$$c_n = \left(\frac{\pi}{2\omega}\right)^{\frac{1}{4}} \frac{\chi_0^n \frac{n!}{2!}}{(-1)^{\frac{n}{2}} n!} e^{-\frac{1}{4}|\chi_0|^2}. \quad (54)$$

In general, the quantum solutions do not offer semiclassical description of some space-time region unless we introduce a decoherence mechanism which is widely regarded as necessary to assign a probability for the occurrence of a classical space-time. However, in the lack of desired decoherence mechanism, in order for a satisfactory classical-quantum correspondence is achieved we may at least investigate if the absolute values of the solutions of Wheeler-DeWitt equation have maxima in the vicinity of the classical loci. Fig.7 (left) shows the square of the wavefunction in the case of negative cosmological constant. As we can see from this figure and the corresponding classical trajectory in Fig.6 (left), the peak follows the classical trajectory. Thus, it is seen that there is a good correspondence between quantum description and the classical pattern of the model.

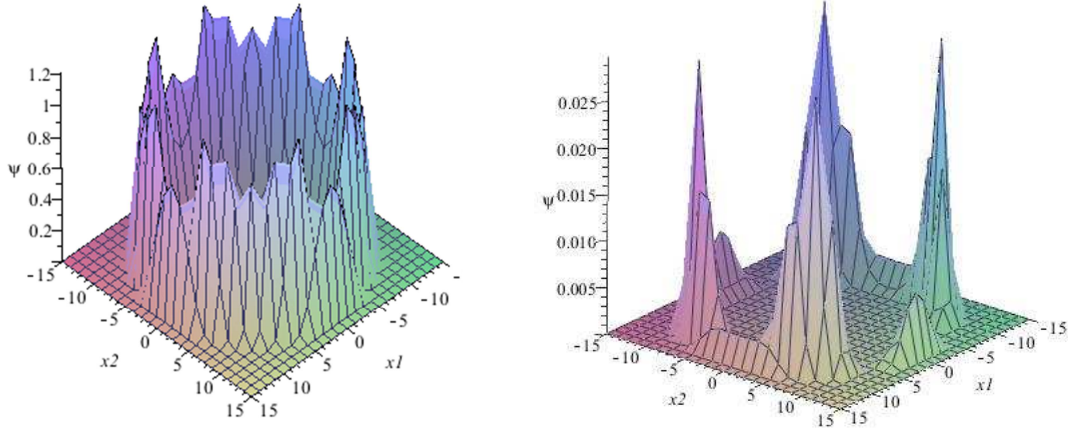


Figure 7: The square of the wavefunction of the universe in  $x_1 - x_2$  plane with one extra dimension and negative cosmological constant. Left and right figures refer to the commutative and GUP framework, respectively.

We get the corresponding answers in the positive cosmological background by replacing  $\omega^2$  with  $-\omega^2$  in the above equations. Fig.7, shows the square of the wavefunction. Because of highly increasing dependence of the amplitude of the wavefunction on  $x_1$  and  $x_2$ , the fluctuation of wavefunction on the line  $x_1 = x_2$  can not be seen using the normal scales adopted in this figure. However, the peak of squared wavefunction follows properly the classical trajectory, namely the line  $x_1 = x_2$  in Fig.6 (right). Therefore, similar to the case of negative cosmological constant, there is a good correspondence between quantum description and the classical pattern.

## 4.2 GUP case

Here, we aim to investigate influence of GUP in quantum cosmological model that has been presented above. The Hamiltonian of the model is given by (12). To construct WD equation in GUP framework we use Eq.(36) with the representation  $p_{i0} = -i\frac{\partial}{\partial x_{i0}}$  for the momentum operator. In first order in  $\beta$ , by ignoring zero subscript we have

$$\left[ 2\beta \left( \frac{\partial^4}{\partial x_1^4} + \frac{\partial^4}{\partial x_2^4} - 2\frac{\partial^2}{\partial x_1^2} \frac{\partial^2}{\partial x_2^2} \right) + \left( -\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) + 4\omega^2(x_1^2 - x_2^2) \right] \Psi(x_1, x_2) = 0. \quad (55)$$

We see that this is not a separable differential equation, so we set a procedure to convert it into a separable one. Up to first order in  $\beta$ , we assume a perturbative solution,  $\Psi = \Psi^{(0)} + \beta\Psi^{(1)}$ , where  $\Psi^{(0)}$  is the wave function of commutative case, namely the solution of Eq.(47). Substitution of this perturbative solution into Eq.(55) reveals that due to the presence of  $\beta$  in the first order part of Eq.(55) we may put  $\Psi \simeq \Psi^{(0)}$ , hence we can use the commutative case equation (47) for this part. By this consideration we have

$$\left[ 2\beta \left( \frac{\partial^4}{\partial x_1^4} + \frac{\partial^4}{\partial x_2^4} - 2\frac{\partial^2}{\partial x_2^2} \left( \frac{\partial^2}{\partial x_2^2} + 4\omega^2(x_1^2 - x_2^2) \right) \right) + \left( -\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) + 4\omega^2(x_1^2 - x_2^2) \right] \Psi(x_1, x_2) = 0, \quad (56)$$

$$\left[ 2\beta \left( \frac{\partial^4}{\partial x_1^4} - \frac{\partial^4}{\partial x_2^4} - 8\omega^2 x_1^2 \frac{\partial^2}{\partial x_2^2} + 8\omega^2 \frac{\partial^2}{\partial x_2^2} (x_2^2) \right) + \left( -\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) + 4\omega^2 (x_1^2 - x_2^2) \right] \Psi(x_1, x_2) = 0. \quad (57)$$

Because of the term  $x_1^2 \frac{\partial^2}{\partial x_2^2}$ , the equation is inseparable too. To resolve the problem we use the classical loci  $x_1^2 + x_2^2 = r^2$  (see figure 6).

$$\left[ 2\beta \left( \frac{\partial^4}{\partial x_1^4} - \frac{\partial^4}{\partial x_2^4} - 8\omega^2 (r^2 - x_2^2) \frac{\partial^2}{\partial x_2^2} + 8\omega^2 \frac{\partial^2}{\partial x_2^2} (x_2^2) \right) + \left( -\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) + 4\omega^2 (x_1^2 - x_2^2) \right] \Psi = 0.$$

Separation of the variables as  $\Psi(x_1, x_2) = U(x_1)V(x_2)$  with perturbative solutions in first order of  $\beta$  as  $U(x_1) = U^{(0)}(x_1) + \beta U^{(1)}(x_1)$  and  $V(x_2) = V^{(0)}(x_2) + \beta V^{(1)}(x_2)$ , and perturbative separation of the separation constant  $\lambda = \lambda^{(0)} + \beta \lambda^{(1)}$  leads to

- in zeroth order in  $\beta$

$$\frac{\partial^2 W^{(0)}(x_i)}{\partial x_i^2} + (\lambda^{(0)} - 4\omega^2 x_i^2) W^{(0)}(x_i) = 0, \quad (58)$$

$$W^{(0)}(x_i; i = 1, 2) = U^{(0)}(x_1), V^{(0)}(x_2),$$

- in first order in  $\beta$ .

$$\frac{\partial^2 W^{(1)}(x_i)}{\partial x_i^2} + (\lambda^{(0)} - 4\omega^2 x_i^2) W^{(1)}(x_i) = g(x_i), \quad (59)$$

$$W^{(1)}(x_i; i = 1, 2) = U^{(1)}(x_1), V^{(1)}(x_2),$$

$$g(x_1) = 2 \frac{\partial^4 U^{(0)}(x_1)}{\partial x_1^4} - \lambda^{(1)} U^{(0)}(x_1),$$

$$g(x_2) = 2 \left( \frac{\partial^4 V^{(0)}}{\partial x_2^4} + 8\omega^2 (r^2 - x_2^2) \frac{\partial^2 V^{(0)}}{\partial x_2^2} - 8\omega^2 \frac{\partial^2 (x_2^2 V^{(0)})}{\partial x_2^2} \right) - \lambda^{(1)} V^{(0)}.$$

We can see that Eq.(58) is commutative limit ( $\beta \rightarrow 0$ ) of Eq.(55) (see Eq.(49)). So it's solution is

$$W_n^{(0)}(x_i) = \left( \frac{2\omega}{\pi} \right)^{1/4} \frac{e^{-\omega x_i^2}}{\sqrt{2^n n!}} H_n(\sqrt{2\omega} x_i). \quad (60)$$

Eq.(59) is a inhomogeneous differential equation. The solution of homogeneous part is



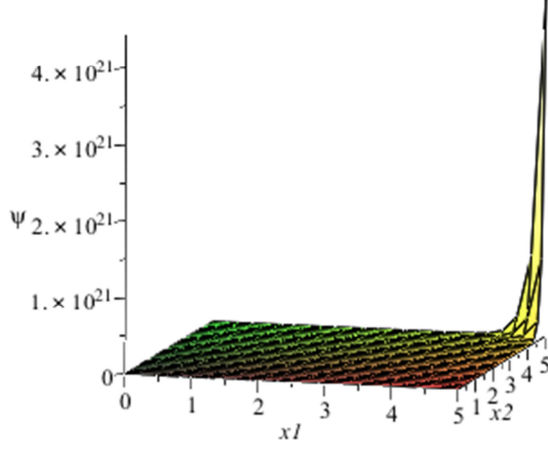


Figure 8: The square of the wavefunction of universe in  $x_1 - x_2$  plane with one extra dimension and positive cosmological constant.

$$\begin{aligned} W_n^{(1)}(H) &= \left( \frac{2\omega}{\pi} \right)^{1/4} \frac{e^{-\omega x_i^2}}{\sqrt{2^n n!}} H_n(\sqrt{2\omega} x_i) \\ &= W_n^{(0)}(x_i). \end{aligned} \quad (61)$$

By use of Eq.(61), the solution of inhomogeneous part can be written as

$$W_n^{(1)}(I) = \xi_1(x_i)W_0^{(0)}(x_i) + \xi_2(x_i)W_n^{(0)}(x_i), \quad (62)$$

where

$$\begin{aligned} \xi_1(x_i) &= \frac{1}{\mu(x_i)} \int^{x_i} \mu(x_i) \frac{g_n W_n^{(0)}}{W_0^{(0)'} W_n^{(0)} - W_n^{(0)'} W_0^{(0)}} dx_i, \\ \xi_2(x_i) &= \int^{x_i} \frac{2n W_0^{(0)2} \xi_1 - g_n W_0^{(0)}}{W_0^{(0)'} W_n^{(0)} - W_n^{(0)'} W_0^{(0)}} dx_i, \\ \mu(x_i) &= \text{Exp} \left( \int^{x_i} \frac{2n W_0^{(0)} W_n^{(0)}}{W_0^{(0)'} W_n^{(0)} - W_n^{(0)'} W_0^{(0)}} dx_i \right), \end{aligned} \quad (63)$$

and a prime denotes for differentiation with respect to  $x_i$ . Combination of zeroth and first order solutions leads to

$$W_n(x_i) = W_n^{(0)}(x_i) + \beta(W_n^{(0)}(x_i) + \xi_1(x_i)W_0^{(0)}(x_i) + \xi_2(x_i)W_n^{(0)}(x_i)). \quad (64)$$

Therefore, we may write the general solution of WD equation in the GUP framework as a superposition of above eigenfunctions

$$\Psi_{GUP}(x_1, x_2) = \sum_n c_n W_n(x_1) W_n(x_2). \quad (65)$$

We set, as in Eq.(54), the coefficient  $c_n$  to the coefficient of commutative wavefunction. Figure.7 (right) shows the square of the wavefunction with negative cosmological constant

in GUP framework. As is seen in this figure, there are four peaks distributed around  $x_1, x_2 = \pm 8.5$  which in the classical viewpoint correspond to a state with  $R_{max}$ . Also, the overall peak follows the classical trajectory in figure.6 (left). Thus, similar to the commutative case, there is a good correspondence between quantum description and the classical pattern of the model in GUP case.

## 5 Problem of time

The canonical approach at the classical level gives the Hamiltonian constraint. Promoting this constraint to the quantum level does not give a time-dependent wave equation, rather leads to a stationary and timeless equation so called Wheeler-DeWitt equation. This suggests that nothing at all evolves in the universe. Therefore, one is faced with the problem of explaining the origin of the notions of time in the laws of physics. This is so called time problem in quantum gravity and there are different approaches to solve this problem [26]. Among these approaches, a new and relevant approach to this paper is the one introduced in [27]-[28] where the authors offer an explanation of time evolution in quantum gravity by using of the probability density in quantum mechanics. They establish a method called "Probabilistic Evolutionary Process" (PEP) which states that if the system is in a given quantum state, then a small perturbation inspires this state to end up in another quantum state with the higher probability of existence. The evolutionary process of initial quantum states toward the more probable final quantum states suggests a possible solution to the problem of time in quantum cosmology.

In our problem by consideration of time evolution in classical context, which accords with increasing in  $x_1$  and  $x_2$ , we may say that such a transition in quantum state from low probability to high probability, can be considered as a quantum evolutionary process. This interpretation of time evolution may be justified in the positive cosmological constant case, where there is considerable difference in the probability amplitude of wavefunction. However, because of nearly equal probability amplitude of wavefunction in the negative cosmological constant case, this interpretation of time evolution is not of practical use to solve the time problem.

In classical context, the time evolution is in accordance with going from  $R_{min}$  to  $R_{max}$ . As we mentioned in previous subsection, by consideration of higher probability of existence in quantum mechanics and using the discussion in references [27]-[28], we may explain the time problem of the WD equation in quantum gravity. It should be noted that, this interpretation is due to the influence of GUP.

## 6 Conclusion

We have considered a multidimensional cosmology having FRW type metric of 4-dimensional space-time and  $d$ -dimensional Ricci-flat internal space coupled with a higher dimensional cosmological constant. The classical cosmology in commutative and GUP cases are studied in detail and the corresponding exact solutions for negative and positive cosmological constants are obtained. For negative cosmological constant, both cases result in finite size universes which differ by their size and age, while, for positive cosmological constant

both cases result in infinite size universes having late time accelerating behavior in good agreement with the current observations. Both commutative and GUP cases with negative and positive cosmological constants result in the stabilization of internal space to the sub-Planck size. We have also derived the Wheeler-DeWitt equation and obtained the solutions in both cases for both negative and positive cosmological constants. It is shown that good correspondence exists between the classical and quantum solutions.

It is worth noting that the effects of GUP in this model are important not only in the early universe but also in the late time behavior of the cosmic evolution. So, in the GUP framework, we can see somehow the indirect quantum gravitational effects at large scale. Moreover, the influence of GUP to solve the time problem in this model is remarkable.

## 7 Appendix

We want to show that the momentum  $p_j$  in the GUP framework in terms of the ordinary low energy momentum  $p_{j0}$ , equation (35) fulfill the commutation relations (32) and (33) with  $\beta' = 2\beta$  up to first order in  $\beta$ . Let us compute (32)

$$\begin{aligned}
[x_i, p_j] &= [x_{i0}, p_{j0} + \beta p_{j0} p_{k0} p_{k0}] = [x_{i0}, p_{j0}] + \beta [x_{i0}, p_{j0} p_{k0} p_{k0}] \\
&= i\delta_{ij} + \beta p_{j0} [x_{i0}, p_{k0} p_{k0}] + \beta [x_{i0}, p_{j0}] p_{k0} p_{k0} \\
&= i\delta_{ij} + 2i\beta \delta_{ik} p_{k0} p_{j0} + i\beta \delta_{ij} p_{k0} p_{k0} \\
&= i(\delta_{ij} + \beta \delta_{ij} p_0^2 + 2\beta p_{i0} p_{j0}),
\end{aligned} \tag{66}$$

where we have used the identity  $[P, QS] = Q[P, S] + [P, Q]S$ . Up to first order in  $\beta$  it is obvious that

$$\beta p^2 = \beta p_i p_i = \beta p_{i0} + \beta p_{i0} p_{k0} p_{k0} = \beta p_{i0} p_{i0} = \beta p_0^2, \tag{67}$$

and similarly  $\beta p_i p_j = \beta p_i p_j$ . Substituting these in equation (66) yields

$$[x_i, p_j] = i(\delta_{ij} + \beta \delta_{ij} p^2 + 2\beta p_i p_j), \tag{68}$$

which is nothing but Eq.(32) with  $\beta' = 2\beta$ . Eq. (33) is a direct consequence of  $[p_{i0}, p_{j0}] = 0$ .

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